

Asymptotic expansions of solutions of fractional diffusion equations

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Abstract

In this paper we obtain the precise description of the asymptotic behavior of the solution u of

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty), \quad u(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}^N,$$

where $0 < \theta < 2$ and $\varphi \in L_K := L^1(\mathbf{R}^N, (1 + |x|)^K dx)$ with $K \geq 0$. Furthermore, we develop the arguments in [15] and [18] and establish a method to obtain the asymptotic expansions of the solutions to a nonlinear fractional diffusion equation

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = |u|^{p-1} u \quad \text{in } \mathbf{R}^N \times (0, \infty),$$

where $0 < \theta < 2$ and $p > 1 + \theta/N$.

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1 Introduction

This paper is concerned with the Cauchy problem for a nonlinear fractional diffusion equation

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = |u|^{p-1} u \quad \text{in } \mathbf{R}^N \times (0, \infty), \quad u(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}^N, \quad (1.1)$$

where $N \geq 1$, $\partial_t := \partial/\partial t$, $(-\Delta)^{\theta/2}$ is the fractional power of the Laplace operator with $0 < \theta < 2$, $p > 1 + \theta/N$ and $\varphi \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$. We say that a continuous function u in $\mathbf{R}^N \times (0, \infty)$ is a solution of (1.1) if u satisfies

$$u(x, t) = \int_{\mathbf{R}^N} G_\theta(x - y, t) \varphi(y) dy + \int_0^t \int_{\mathbf{R}^N} G_\theta(x - y, t - s) F(u(y, s)) dy ds \quad (1.2)$$

for $x \in \mathbf{R}^N$ and $t > 0$, where $F(u) := |u|^{p-1}u$. Here $G_\theta = G_\theta(x, t)$ is the fundamental solution of the linear fractional diffusion equation

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty). \quad (1.3)$$

Problem (1.1) appears in the study of nonlinear problems with anomalous diffusion and the Laplace equation with a dynamical boundary condition and it has been studied extensively by many mathematicians (see [1], [6], [7], [10], [11], [14], [17], [18], [24] and references therein). Among others, Sugitani [24] showed that, if $1 < p \leq 1 + \theta/N$, then problem (1.1) possesses no positive global in time solutions. On the other hand, if $p > 1 + \theta/N$ and $\|\varphi\|_{L^{N(p-1)/\theta, \infty}(\mathbf{R}^N)}$ is sufficiently small, then there exists a global in time solution u of (1.1) such that

$$\lim_{t \rightarrow \infty} t^{\frac{N}{\theta}(1-\frac{1}{q})} \|u(t) - m_u G_\theta(t)\|_{L^q(\mathbf{R}^N)} = 0 \quad (1.4)$$

for any $1 \leq q \leq \infty$, where

$$m_u := \lim_{t \rightarrow \infty} \int_{\mathbf{R}^N} u(x, t) dx.$$

(See [17] and [18].) In this paper we investigate the higher order asymptotic expansions of the solutions to (1.1) satisfying (1.4).

Let us consider the Cauchy problem for a nonlinear heat equation

$$\partial_t v - \Delta v = f(x, t, v, \nabla v) \quad \text{in } \mathbf{R}^N \times (0, \infty), \quad v(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}^N, \quad (1.5)$$

where f is a continuous function in $\mathbf{R}^N \times (0, \infty) \times \mathbf{R}^{N+1}$ and

$$\varphi \in L_K := L^1(\mathbf{R}^N, (1 + |x|)^K dx) \quad \text{for some } K \geq 0.$$

If a classical solution v of (1.5) satisfies

$$|f(x, t, v(x, t), (\nabla v)(x, t))| \leq C(1 + t)^{-A} \left[|v(x, t)| + (1 + t)^{\frac{1}{2}} |(\nabla v)(x, t)| \right]$$

in $\mathbf{R}^N \times (0, \infty)$ for some $C > 0$ and $A > 1$, then the solution v behaves like a suitable multiple of the Gauss kernel as $t \rightarrow \infty$. More precisely,

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{q})} \|v(t) - m_v \Gamma(t)\|_{L^q(\mathbf{R}^N)} = 0 \quad (1.6)$$

holds for any $1 \leq q \leq \infty$, where

$$\Gamma(x, t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad m_v := \lim_{t \rightarrow \infty} \int_{\mathbf{R}^N} v(x, t) dx.$$

See e.g., [16, Theorem 3.1]. Generally, the higher order asymptotic expansions of the solutions depend on the decay of the solutions at the space infinity and they have been studied in many papers, see e.g., [5], [8], [10], [12], [14], [15], [16], [21] and [25]. Among others, the first and the second authors of this paper developed the arguments in [12] and established a systematic method to obtain the higher order asymptotic expansions of the solutions satisfying (1.6) (see [15] and [16]). Their arguments are based on the fact that $v(t) \in L_K$ for any $t > 0$.

We consider fractional diffusion equations. Let $0 < \theta < 2$ and set

$$[S_\theta(t)\varphi](x) := \int_{\mathbf{R}^N} G_\theta(x - y, t)\varphi(y) dy, \quad x \in \mathbf{R}^N, t > 0.$$

Since $G_\theta(\cdot, t) \in L_K$ for $t > 0$ if and only if $K < \theta$, we see that

$$\{S_\theta(t)\varphi : \varphi \in L_K\} \subset L_K \quad \text{for any } t > 0 \quad \text{if and only if } K < \theta.$$

This means that, in the case $K \geq \theta$, L_K is not a suitable function space of initial functions for the fractional heat semigroup $\{S_\theta(t)\}_{t \geq 0}$. As far as we know, there are no systematic methods which are available to the study of the asymptotic expansions of the solutions to (1.1) with $\varphi \in L_K$ in the case $K \geq \theta$. For the case $0 \leq K < \theta$, see [18]. (For related results, see e.g., [20], [22], [26], [27] and references therein.)

In this paper we refine and develop the arguments in [18] and establish a systematic method to obtain the asymptotic expansions of the solutions to (1.1) without any restrictions such as $K < \theta$. More precisely, the purposes of this paper are the following:

Let $0 < \theta < 2$, $j \in \{0, 1, 2, \dots\}$ and $K \geq 0$.

- (i) We obtain the precise description of the asymptotic behavior of $\nabla^j[S_\theta(t)\varphi]$ with $\varphi \in L_K$ and identify the largest subspace X of $L^1(\mathbf{R}^N)$ satisfying

- $X \subset L_K$,
- $\{\nabla^j[S_\theta(t)\varphi] : \varphi \in X\} \subset X$ for any $t > 0$.

See Theorem 1.1 and Theorem 1.2 (a);

- (ii) We find a subspace Y of $L^1(\mathbf{R}^N)$ such that

- $L_K \subset Y$ and the dimension of the quotient space Y/L_K is finite,
- $\{\nabla^j[S_\theta(t)\varphi] : \varphi \in Y\} \subset Y$ for any $t > 0$.

See Theorem 1.2 (b);

- (iii) Let ψ be a radially symmetric smooth function in \mathbf{R}^N such that $\partial_x^\alpha \psi \in L^\infty(\mathbf{R}^N) \cap L_{K+|\alpha|}$ for $\alpha \in \mathbf{M}_K$. By using $S_\theta(t)\psi$ and its derivatives, we obtain the precise description of the asymptotic behavior of the solution u of (1.1) satisfying (1.4). See Sections 3 and 4.

We introduce some notation. For any $k \geq 0$, let $[k] \in \mathbf{N} \cup \{0\}$ be such that $k-1 < [k] \leq k$. For any multi-index $\alpha \in \mathbf{M} := (\mathbf{N} \cup \{0\})^N$, set

$$|\alpha| := \sum_{i=1}^N \alpha_i, \quad \alpha! := \prod_{i=1}^N \alpha_i!, \quad x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}, \quad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

Let $\mathbf{M}_k := \{\alpha \in \mathbf{M} : |\alpha| \leq k\}$ if $k \geq 0$ and $\mathbf{M}_k := \emptyset$ if $k < 0$. For any $\alpha = (\alpha_1, \dots, \alpha_N)$, $\beta = (\beta_1, \dots, \beta_N) \in \mathbf{M}$, we say $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i \in \{1, \dots, N\}$. Let $\nabla := (\partial/\partial x_1, \dots, \partial/\partial x_N)$. For any $\alpha \in \mathbf{M}$ and $0 < \theta < 2$, we define

$$g_{\alpha,\theta}(x,t) := \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha G_\theta)(x,t) = \frac{(-1)^{|\alpha|}}{\alpha!} t^{-\frac{N+|\alpha|}{\theta}} (\partial_x^\alpha G_\theta)(t^{-\frac{1}{\theta}}x, 1). \quad (1.7)$$

(See (G)-(i) in Section 2.) For any $1 \leq q \leq \infty$, let $\|\cdot\|_q$ be the usual norm of $L^q := L^q(\mathbf{R}^N)$. Furthermore, for any $\ell \geq 0$, we set

$$|||f|||_\ell := \int_{\mathbf{R}^N} |x|^\ell |f(x)| dx, \quad f \in L_\ell.$$

For a real vector space V and a finite set $\{f_i\}_{i=1}^m$ in V , let

$$\sum_{i=1}^m \langle f_i \rangle := \left\{ \sum_{i=1}^m c_i f_i : \{c_i\}_{i=1}^m \subset \mathbf{R} \right\} \subset V.$$

Now we are ready to state the main results of this paper. In Theorem 1.1 we obtain the precise description of the asymptotic behavior of $S_\theta(t)\varphi$ with $\varphi \in L_K$.

Theorem 1.1 *Let $N \geq 1$, $0 < \theta < 2$, $j \in \{0, 1, 2, \dots\}$ and $K \geq 0$. For any $\varphi \in L_K$, set*

$$v(x,t) := [S_\theta(t)\varphi](x) - \sum_{|\alpha| \leq K} \left[\int_{\mathbf{R}^N} x^\alpha \varphi(x) dx \right] g_{\alpha,\theta}(x,t). \quad (1.8)$$

Then

$$(\nabla^j v)(t) \in \left\{ f \in L_K : \int_{\mathbf{R}^N} x^\alpha f(x) dx = 0 \text{ for all } \alpha \in \mathbf{M}_K \right\} \quad (1.9)$$

for $t > 0$. Furthermore, the following hold:

(a) *There exists a constant C such that*

$$t^{\frac{N}{\theta}(1-\frac{1}{q})} \|(\nabla^j v)(t)\|_q + t^{-\frac{\ell}{\theta}} \| |(\nabla^j v)(t)| \|_\ell \leq C t^{-\frac{K+j}{\theta}} \| |\varphi| \|_K, \quad t > 0, \quad (1.10)$$

for $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$. Here C depends only on N , θ , j and K ;

(b) *For any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$,*

$$\lim_{t \rightarrow \infty} t^{\frac{K+j}{\theta}} \left[t^{\frac{N}{\theta}(1-\frac{1}{q})} \|(\nabla^j v)(t)\|_q + t^{-\frac{\ell}{\theta}} \| |(\nabla^j v)(t)| \|_\ell \right] = 0.$$

Remark 1.1 (i) *Theorem 1.1 in the case $0 \leq K < \theta$ has been already proved in [18]. However, the proof in this paper is simpler than that of [18].*

(ii) *In the case $K \geq \theta$, some weaker estimates than (1.10) were stated in [26] and [27]. The proofs in [26] and [27] were based on uncertain pointwise estimates of $(\partial_t^m \partial_x^\alpha G_\theta)(x, t)$, where $m \in \{1, 2, \dots\}$ and $\alpha \in \mathbf{M}$.*

Next we characterize the behavior of $S_\theta(t)\varphi$ by the use of subspaces of L^1 .

Theorem 1.2 *Let $N \geq 1$, $0 < \theta < 2$, $j \in \{0, 1, 2, \dots\}$ and $K \geq 0$.*

(a) *The real vector space*

$$\mathcal{L}_{K,j}^0 := \left\{ f \in L_K : \int_{\mathbf{R}^N} x^\alpha f(x) dx = 0 \quad \text{for } \alpha \in \mathbf{M}_{K-j-\theta} \right\}$$

is the largest function space satisfying

$$\mathcal{L}_{K,j}^0 \subset L_K \quad \text{and} \quad \bigcup_{t>0} \{ \nabla^j [S_\theta(t)\varphi] : \varphi \in \mathcal{L}_{K,j}^0 \} \subset \mathcal{L}_{K,j}^0. \quad (1.11)$$

(b) *Define*

$$\mathcal{L}_{K,j} := L_K + \sum_{|\alpha|+m\theta \leq K-j} \langle (\partial_t^m \partial_x^\alpha G_\theta)(\cdot, 1) \rangle.$$

Then

$$\bigcup_{t>0} \{ \nabla^j [S_\theta(t)\varphi] : \varphi \in \mathcal{L}_{K,j} \} \subset \mathcal{L}_{K,j} \subset L_{K'},$$

where $0 \leq K' \leq K$ with $K' < \theta$.

Assertion (b) implies that the dimension of the quotient space $\mathcal{L}_{K,j}/L_K$ is finite. Furthermore, as an application of Theorem 1.1, we study the behavior of the hot spots of $S_\theta(t)\varphi$.

Theorem 1.3 *Let $N \geq 1$, $0 < \theta < 2$ and $K \geq 2$. Let $\varphi \in L_K$ and assume that*

$$M(\varphi) := \int_{\mathbf{R}^N} \varphi(x) dx > 0.$$

Then the hot spots of $S_\theta(t)\varphi$

$$H(t) := \left\{ x \in \mathbf{R}^N : [S_\theta(t)\varphi](x) = \sup_{y \in \mathbf{R}^N} [S_\theta(t)\varphi](y) \right\}$$

is not empty for any $t > 0$ and the following hold:

- *$H(t)$ consists of only one point $\{x(t)\}$ in finite time;*
- *$x(t)$ moves along a smooth curve in \mathbf{R}^N and*

$$\lim_{t \rightarrow \infty} x(t) = C(\varphi) := \int_{\mathbf{R}^N} x\varphi(x) dx \Big/ M(\varphi).$$

The movement of the hot spots is of independent interest and it has been studied for the heat equation, see [4], [9], [13], [19] and references therein.

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.1 by refining the arguments in [12] and [18]. Theorems 1.2 and 1.3 follow from Theorem 1.1. In Section 3 we obtain the asymptotic expansions of the solutions to the fractional diffusion equation with an inhomogeneous term

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = F(x, t) \quad \text{in } \mathbf{R}^N \times (0, \infty), \quad u(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}^N,$$

where $\varphi \in L^\infty \cap L_K$ for some $K \geq 0$ and $F \in L^\infty(0, T; L_K)$ for any $T > 0$. In Section 4 we apply the arguments in Sections 2 and 3 to obtain the asymptotic expansions of the solution of (1.1) satisfying (1.4). Furthermore, combining the arguments in [15], we obtain the higher order asymptotic expansions of the solution u of (1.1).

2 Proofs of Theorems 1.1, 1.2 and 1.3

In this section we recall some properties of the fundamental solution $G_\theta = G_\theta(x, t)$ and prove Theorems 1.1, 1.2 and 1.3.

Let $0 < \theta < 2$. The fundamental solution $G_\theta = G_\theta(x, t)$ is represented by

$$G_\theta(x, t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{ix \cdot \xi} e^{-t|\xi|^\theta} d\xi \tag{2.1}$$

and it has the following properties:

(G) $G_\theta = G_\theta(x, t)$ is a positive smooth function in $\mathbf{R}^N \times (0, \infty)$ such that

- (i) $G_\theta(x, t) = t^{-\frac{N}{\theta}} G_\theta(t^{-\frac{1}{\theta}} x, 1)$ for $x \in \mathbf{R}^N$ and $t > 0$;
- (ii) $\sup_{x \in \mathbf{R}^N} (1 + |x|)^{N+\theta+|\alpha|} |(\partial_x^\alpha G_\theta)(x, 1)| < \infty$ for $\alpha \in \mathbf{M}$;
- (iii) $G_\theta(\cdot, 1)$ is radially symmetric, monotone decreasing with respect to $r := |x|$ and

$$\liminf_{|x| \rightarrow +\infty} (1 + |x|)^{N+\theta+j} (\partial_r^j G_\theta)(x, 1) > 0$$

for $j \in \{0, 1, 2, \dots\}$;

$$(iv) \quad G_\theta(x, t) = \int_{\mathbf{R}^N} G_\theta(x - y, t - s) G_\theta(y, s) dy \quad \text{for } x \in \mathbf{R}^N \text{ and } t > s > 0.$$

See [2] and [3]. (See also [17], [18] and [24].) By (G)-(i) and (ii), for any $\alpha \in \mathbf{M}$, we can find a positive constant C_α such that

$$|(\partial_x^\alpha G_\theta)(x, t)| \leq C_\alpha t^{-\frac{N+|\alpha|}{\theta}} (1 + t^{-\frac{1}{\theta}} |x|)^{-N-\theta-|\alpha|} \quad (2.2)$$

for $x \in \mathbf{R}^N$ and $t > 0$. This implies that

$$\sup_{t>0} \left[t^{\frac{N}{\theta}(1-\frac{1}{q})+\frac{|\alpha|}{\theta}} \|(\partial_x^\alpha G_\theta)(t)\|_q + t^{\frac{|\alpha|-\ell}{\theta}} \|(\partial_x^\alpha G_\theta)(t)\|_\ell \right] < \infty \quad (2.3)$$

for $1 \leq q \leq \infty$, $\alpha \in \mathbf{M}$ and $\ell \in [0, \theta + |\alpha|]$. Furthermore, for any $j \in \{0, 1, 2, \dots\}$, by the Young inequality and (2.3) we can find a positive constant C_j such that

$$\|\nabla^j [S_\theta(t)\varphi]\|_r \leq C_j t^{-\frac{N}{\theta}(\frac{1}{q}-\frac{1}{r})-\frac{j}{\theta}} \|\varphi\|_q, \quad t > 0, \quad (2.4)$$

for $\varphi \in L^q$ and $1 \leq q \leq r \leq \infty$.

Next we state a lemma on pointwise estimates of $(\partial_t^m \partial_x^\alpha G_\theta)(x, t)$, where $\alpha \in \mathbf{M}$ and $m \in \{1, 2, \dots\}$. In what follows, by the letter C we denote generic positive constants (independent of x and t) and they may have different values also within the same line.

Lemma 2.1 *Let $\alpha \in \mathbf{M}$ and $m \in \{1, 2, \dots\}$. There exists a positive constant C such that*

$$|(\partial_t^m \partial_x^\alpha G_\theta)(x, t)| \leq C t^{-\frac{N+|\alpha|}{\theta}-m} (1 + t^{-\frac{1}{\theta}} |x|)^{-N-\theta m-|\alpha|}, \quad x \in \mathbf{R}^N, \quad t > 0. \quad (2.5)$$

Proof. It follows from (2.1) that

$$|(\partial_t^m \partial_x^\alpha G_\theta)(x, t)| = t^{-\frac{N+|\alpha|}{\theta}-m} \left| \mathcal{F}^{-1}[f](t^{-\frac{1}{\theta}} x) \right|, \quad x \in \mathbf{R}^N, \quad t > 0, \quad (2.6)$$

where $f(\xi) := \xi^\alpha |\xi|^{\theta m} e^{-|\xi|^\theta}$. Then we have

$$\partial_\xi^\beta f \in L^1(\mathbf{R}_\xi^N) \quad \text{for all } \beta \in \mathbf{M}_\Lambda,$$

$$|\partial_\xi^\beta f(\xi)| \leq C |\xi|^{\theta m + |\alpha| - |\beta|} \quad \text{for all } \xi \in \mathbf{R}^N \setminus \{0\} \text{ and } \beta \in \mathbf{M}_{\Lambda+1},$$

where $\Lambda := |\alpha| + [\theta m] + N - 1 \in \{0, 1, 2, \dots\}$. By the Hörmander-Mikhlin type multiplier theorem (see [23]) we obtain

$$|\mathcal{F}^{-1}[f](x)| \leq C|x|^{-N-\theta m-|\alpha|}, \quad x \in \mathbf{R}^N. \quad (2.7)$$

On the other hand, it follows that

$$|\mathcal{F}^{-1}[f](x)| \leq C\|f\|_1 \leq C, \quad x \in \mathbf{R}^N. \quad (2.8)$$

We deduce from (2.7) and (2.8) that

$$|\mathcal{F}^{-1}[f](x)| \leq C(1 + |x|)^{-N-\theta m-|\alpha|}, \quad x \in \mathbf{R}^N.$$

This together with (2.6) implies (2.5), and Lemma 2.1 follows. \square

We prepare the following lemma for the proof of Theorem 1.1.

Lemma 2.2 *Let $j \in \{0, 1, 2, \dots\}$ and $\ell \geq 0$. For any $x, y \in \mathbf{R}^N$ and $t > 0$, set*

$$H_\ell^j(x, y, t) := (\nabla^j G_\theta)(x - y, t) - \sum_{|\alpha| \leq [\ell]} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha \nabla^j G_\theta)(x, t) y^\alpha.$$

(a) *There exists $C_1 > 0$ such that*

$$\int_{\mathbf{R}^N} |x|^\ell |H_\ell^j(x, y, t)| dx \leq C_1 t^{-\frac{j}{\theta}} |y|^\ell \quad (2.9)$$

for $y \in \mathbf{R}^N$ and $t > 0$.

(b) *There exists $C_2 > 0$ such that*

$$\int_{\mathbf{R}^N} |x|^\ell \left(\int_{\mathbf{R}^N} |H_K^j(x, y, t)| |\varphi(y)| dy \right) dx \leq C_2 t^{-\frac{K+j-\ell}{\theta}} |||\varphi|||_K, \quad t > 0,$$

for $\varphi \in L_K$ and $0 \leq \ell \leq K$.

(c) *For any $\varphi \in L_K$ and $0 \leq \ell \leq K$,*

$$\lim_{t \rightarrow \infty} t^{\frac{K+j-\ell}{\theta}} \int_{\mathbf{R}^N} |x|^\ell \left| \int_{\mathbf{R}^N} H_K^j(x, y, t) \varphi(y) dy \right| dx = 0.$$

Proof. We prove assertion (a). Let $y \in \mathbf{R}^N$. Since

$$\begin{aligned} H_\ell^j(x, y, t) &= \frac{1}{[\ell]!} \int_0^1 (1-\tau)^{[\ell]} \frac{d^{[\ell]+1}}{d\tau^{[\ell]+1}} (\nabla^j G_\theta)(x - \tau y, t) d\tau \\ &= (-1)^{[\ell]+1} ([\ell] + 1) \sum_{|\alpha| = [\ell]+1} \frac{y^\alpha}{\alpha!} \int_0^1 (1-\tau)^{[\ell]} (\partial_x^\alpha \nabla^j G_\theta)(x - \tau y, t) d\tau, \end{aligned} \quad (2.10)$$

by (2.2) we have

$$\begin{aligned}
& \int_{\{|x| \geq 2|y|\}} |x|^\ell |H_\ell^j(x, y, t)| dx \\
& \leq C \int_0^1 \int_{\{|x| \geq 2|y|\}} |x|^\ell \left| (\nabla^{[\ell]+j+1} G_\theta)(x - \tau y, t) \right| |y|^{[\ell]+1} dx d\tau \\
& \leq C |y|^\ell \int_0^1 \int_{\{|x| \geq 2|y|\}} |x|^{[\ell]+1} \left| (\nabla^{[\ell]+j+1} G_\theta)(x - \tau y, t) \right| dx d\tau \\
& \leq C |y|^\ell \int_0^1 \int_{\{|x| \geq 2|y|\}} |x|^{[\ell]+1} t^{-\frac{N}{\theta} - \frac{[\ell]+j+1}{\theta}} \left(1 + t^{-\frac{1}{\theta}} |x - \tau y| \right)^{-(N+\theta+[\ell]+j+1)} dx d\tau.
\end{aligned} \tag{2.11}$$

It follows that

$$|x - \tau y| \geq |x| - |y| \geq |x|/2 \quad \text{if} \quad |x| \geq 2|y| \quad \text{and} \quad 0 \leq \tau \leq 1.$$

This together with (2.11) implies that

$$\begin{aligned}
& \int_{\{|x| \geq 2|y|\}} |x|^\ell |H_\ell^j(x, y, t)| dx \\
& \leq C |y|^\ell \int_{\mathbf{R}^N} |x|^{[\ell]+1} t^{-\frac{N}{\theta} - \frac{[\ell]+j+1}{\theta}} \left(1 + t^{-\frac{1}{\theta}} \frac{|x|}{2} \right)^{-(N+\theta+[\ell]+j+1)} dx \leq C t^{-\frac{j}{\theta}} |y|^\ell.
\end{aligned} \tag{2.12}$$

On the other hand, by (2.2) we see that

$$\begin{aligned}
& \int_{\{|x| < 2|y|\}} |x|^\ell |H_\ell^j(x, y, t)| dx \\
& \leq \int_{\{|x| < 2|y|\}} |x|^\ell |(\nabla^j G_\theta)(x - y, t)| dx \\
& \quad + C \sum_{|\alpha| \leq [\ell]} \int_{\{|x| < 2|y|\}} |x|^\ell |(\partial_x^\alpha \nabla^j G_\theta)(x, t)| |y|^{|\alpha|} dx \\
& \leq (2|y|)^\ell \int_{\mathbf{R}^N} |(\nabla^j G_\theta)(x, t)| dx + C \sum_{|\alpha| \leq [\ell]} |y|^\ell \int_{\{|x| < 2|y|\}} |x|^{|\alpha|} |(\partial_x^\alpha \nabla^j G_\theta)(x, t)| dx \\
& \leq C t^{-\frac{j}{\theta}} |y|^\ell.
\end{aligned} \tag{2.13}$$

By (2.12) and (2.13) we obtain (2.9). Thus assertion (a) follows.

We prove assertions (b) and (c). It follows from Lemma 2.2 (a) and (2.2) that

$$\begin{aligned}
& \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} |x|^\ell |H_K^j(x, y, t)| dx \right) |\varphi(y)| dy \\
& \leq \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} |x|^\ell |H_\ell^j(x, y, t)| dx \right) |\varphi(y)| dy \\
& \quad + C \sum_{[\ell] < |\alpha| \leq K} \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} |x|^\ell |(\partial_x^\alpha \nabla^j G_\theta)(x, t)| dx \right) |y|^{|\alpha|} |\varphi(y)| dy \\
& \leq Ct^{-\frac{j}{\theta}} \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} |y|^\ell |\varphi(y)| dy + C \sum_{[\ell] < |\alpha| \leq K} t^{-\frac{|\alpha|+j-\ell}{\theta}} \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} |y|^{|\alpha|} |\varphi(y)| dy \quad (2.14) \\
& \leq Ct^{-\frac{j}{\theta}} \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} |y|^\ell \left(\frac{|y|}{R^{\frac{1}{\theta}}} \right)^{K-\ell} |\varphi(y)| dy \\
& \quad + C \sum_{[\ell] < |\alpha| \leq K} t^{-\frac{|\alpha|+j-\ell}{\theta}} \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} \left(\frac{|y|}{R^{\frac{1}{\theta}}} \right)^{K-|\alpha|} |y|^{|\alpha|} |\varphi(y)| dy \\
& = Ct^{-\frac{K+j-\ell}{\theta}} \left[(R^{-1}t)^{\frac{K-\ell}{\theta}} + \sum_{[\ell] < |\alpha| \leq K} (R^{-1}t)^{\frac{K-|\alpha|}{\theta}} \right] \int_{\{|y| \geq R^{\frac{1}{\theta}}\}} |y|^K |\varphi(y)| dy
\end{aligned}$$

for all $R > 0$. On the other hand, by (2.2) and (2.10) we obtain

$$\begin{aligned}
& \int_{\{|y| < R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} |x|^\ell |H_K^j(x, y, t)| dx \right) |\varphi(y)| dy \\
& \leq C \int_0^1 \int_{\{|y| < R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} |x|^\ell |(\nabla^{[K]+j+1} G_\theta)(x - \tau y, t)| |y|^{[K]+1} dx \right) |\varphi(y)| dy d\tau \\
& = C \int_0^1 \int_{\{|y| < R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} |x + \tau y|^\ell |(\nabla^{[K]+j+1} G_\theta)(x, t)| dx \right) |y|^{[K]+1} |\varphi(y)| dy d\tau \quad (2.15) \\
& \leq C \int_{\{|y| < R^{\frac{1}{\theta}}\}} \left(\int_{\mathbf{R}^N} (|x|^\ell + |y|^\ell) |(\nabla^{[K]+j+1} G_\theta)(x, t)| dx \right) |y|^{[K]+1} |\varphi(y)| dy \\
& \leq C \int_{\{|y| < R^{\frac{1}{\theta}}\}} (t^{-\frac{[K]+j+1-\ell}{\theta}} + t^{-\frac{[K]+j+1}{\theta}} |y|^\ell) |y|^{[K]+1} |\varphi(y)| dy \\
& \leq Ct^{-\frac{j}{\theta}} (t^{-\frac{[K]+1-\ell}{\theta}} R^{\frac{[K]+1-K}{\theta}} + t^{-\frac{[K]+1}{\theta}} R^{\frac{[K]+\ell+1-K}{\theta}}) \int_{\{|y| < R^{\frac{1}{\theta}}\}} |y|^K |\varphi(y)| dy
\end{aligned}$$

for all $R > 0$. Then, by (2.14) and (2.15) we set $R = t$ to obtain

$$\int_{\mathbf{R}^N} |x|^\ell \left(\int_{\mathbf{R}^N} |H_K^j(x, y, t)| |\varphi(y)| dy \right) dx \leq Ct^{-\frac{K+j-\ell}{\theta}} |||\varphi|||_K, \quad t > 0.$$

This implies assertion (b). Similarly, setting $R = \epsilon t$ with $0 < \epsilon \leq 1$, we have

$$\begin{aligned} & \left| \int_{\mathbf{R}^N} |x|^\ell \left| \int_{\mathbf{R}^N} H_K^j(x, y, t) \varphi(y) dy \right| dx \right. \\ & \leq C t^{-\frac{K+j-\ell}{\theta}} \left[(\epsilon^{-1})^{\frac{K-\ell}{\theta}} + \sum_{[\ell] < |\alpha| \leq K} (\epsilon^{-1})^{\frac{K-|\alpha|}{\theta}} \right] \int_{\{|y| \geq (\epsilon t)^{\frac{1}{\theta}}\}} |y|^K |\varphi(y)| dy \\ & + C t^{-\frac{K+j-\ell}{\theta}} \left(\epsilon^{\frac{[K]+1-K}{\theta}} + \epsilon^{\frac{[K]+\ell+1-K}{\theta}} \right) |||\varphi|||_K. \end{aligned}$$

This implies that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{\frac{K+j-\ell}{\theta}} \left| \int_{\mathbf{R}^N} |x|^\ell \left| \int_{\mathbf{R}^N} H_K^j(x, y, t) \varphi(y) dy \right| dx \right| \\ & \leq C \left(\epsilon^{\frac{[K]+1-K}{\theta}} + \epsilon^{\frac{[K]+\ell+1-K}{\theta}} \right) |||\varphi|||_K. \end{aligned}$$

Then, since ϵ is arbitrary, we obtain assertion (c). Thus Lemma 2.2 follows. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\varphi \in L_K$ and $j \in \{0, 1, 2, \dots\}$. It follows from (1.8) and (2.10) that

$$\begin{aligned} (\nabla^j v)(x, t) &= \nabla^j [S_\theta(t) \varphi](x) - \sum_{|\alpha| \leq K} \left[\int_{\mathbf{R}^N} y^\alpha \varphi(y) dy \right] (\nabla^j g_{\alpha, \theta})(x, t) \\ &= \int_{\mathbf{R}^N} \left[(\nabla^j G_\theta)(x - y, t) - \sum_{|\alpha| \leq K} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha \nabla^j G_\theta)(x, t) y^\alpha \right] \varphi(y) dy \quad (2.16) \\ &= \int_{\mathbf{R}^N} H_K^j(x, y, t) \varphi(y) dy \end{aligned}$$

for $x \in \mathbf{R}^N$ and $t > 0$. Since

$$(\nabla^j v)(x, t) = \left[S_\theta \left(\frac{t}{2} \right) (\nabla^j v) \left(\cdot, \frac{t}{2} \right) \right] (x),$$

by (2.4) we have

$$t^{\frac{N}{\theta}(1-\frac{1}{q})} \|(\nabla^j v)(t)\|_q + t^{-\frac{\ell}{\theta}} \|(\nabla^j v)(t)\|_\ell \leq C \|(\nabla^j v)(t/2)\|_1 + t^{-\frac{\ell}{\theta}} \|(\nabla^j v)(t)\|_\ell, \quad t > 0.$$

Then Lemma 2.2 (b) and (c) with (2.16) imply assertions (a) and (b), respectively.

It remains to prove (1.9). Let $0 \leq |\alpha| \leq K$. By Lemma 2.2 (b) and (2.16) we apply the Fubini theorem to obtain

$$\begin{aligned} \int_{\mathbf{R}^N} x^\alpha (\nabla^j v)(x, t) dx &= \int_{\mathbf{R}^N} x^\alpha \left[\int_{\mathbf{R}^N} H_K^j(x, y, t) \varphi(y) dy \right] dx \\ &= \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} x^\alpha H_K^j(x, y, t) dx \right) \varphi(y) dy \end{aligned} \quad (2.17)$$

for $t > 0$. On the other hand, it follows that

$$\int_{\mathbf{R}^N} x^\alpha (\partial_x^\beta G_\theta)(x, t) dx = 0 \quad \text{if not } \alpha \geq \beta, \quad (2.18)$$

in the case $|\alpha| < |\beta| + \theta$. Then, by (2.10) and (2.18) we have

$$\begin{aligned} & \int_{\mathbf{R}^N} x^\alpha H_K^j(x, y, t) dx \\ &= (-1)^{[K]+1}([K] + 1) \sum_{|\beta|=[K]+1} \frac{1}{\beta!} \int_{\mathbf{R}^N} x^\alpha \int_0^1 (1 - \tau)^{[K]} (\partial_x^\beta \nabla^j G_\theta)(x - \tau y, t) y^\beta d\tau dx \\ &= (-1)^{[K]+1}([K] + 1) \sum_{|\beta|=[K]+1} \frac{y^\beta}{\beta!} \int_0^1 (1 - \tau)^{[K]} \left(\int_{\mathbf{R}^N} x^\alpha (\partial_x^\beta \nabla^j G_\theta)(x - \tau y, t) dx \right) d\tau = 0 \end{aligned}$$

for $y \in \mathbf{R}^N$ and $t > 0$. This together with (2.17) implies that

$$\int_{\mathbf{R}^N} x^\alpha (\nabla^j v)(x, t) dx = 0, \quad t > 0.$$

Thus (1.9) holds and the proof of Theorem 1.1 is complete. \square

By Theorem 1.1 we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $j \in \{0, 1, 2, \dots\}$ and $\varphi \in \mathcal{L}_{K,j}^0$. By the definition of $\mathcal{L}_{K,j}^0$ if $0 \leq K < \theta + j$, then we see that $\mathcal{L}_{K,j}^0$ and the assertion (a) holds. So it suffices to consider the case $K \geq \theta + j$. By Theorem 1.1 we have

$$\nabla^j [S_\theta(t)\varphi] - \sum_{K-j-\theta < |\alpha| \leq K} \left[\int_{\mathbf{R}^N} y^\alpha \varphi(y) dy \right] (\nabla^j g_{\alpha,\theta})(t) \in \mathcal{L}_{K,j}^0, \quad t > 0.$$

If $|\alpha| > K - j - \theta$, then it follows from (2.3) and (2.18) that $(\nabla^j g_{\alpha,\theta})(t) \in \mathcal{L}_{K,j}^0$ for $t > 0$. Therefore $\mathcal{L}_{K,j}^0$ satisfies (1.11). On the other hand, if $\varphi \in L_K \setminus \mathcal{L}_{K,j}^0$, then it follows from Theorem 1.1 and (G)-(ii), (iii) that $\nabla^j [S_\theta(t)\varphi] \notin L_K$ for any $t > 0$. Thus assertion (a) follows.

We prove assertion (b). By Theorem 1.1 we see that

$$\begin{aligned} \{\nabla^j [S_\theta(t)\varphi] : \varphi \in \mathcal{L}_{K,j}\} &\subset L_K + \sum_{|\alpha| \leq K} \langle (\partial_x^\alpha \nabla^j G_\theta)(\cdot, t) \rangle \\ &\quad + \sum_{|\alpha| + m\theta \leq K - j} \langle (\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(\cdot, 1 + t) \rangle \end{aligned} \quad (2.19)$$

for $t > 0$. For any $\alpha \in \mathbf{M}$ and $m \in \{0, 1, 2, \dots\}$ with $|\alpha| + \theta m \leq K - j$, let m' be the largest nonnegative integer satisfying $|\alpha| + \theta(m + m') \leq K - j$. Since

$$\begin{aligned} (\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(x, t) &= \sum_{i=0}^{m'} \frac{(t-1)^i}{i!} (\partial_t^{m+i} \partial_x^\alpha \nabla^j G_\theta)(x, 1) \\ &\quad + \frac{(t-1)^{m'+1}}{(m')!} \int_0^1 (1-\tau)^{m'} \frac{d^{m'+1}}{d\tau^{m'+1}} [(\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(x, 1 + \tau(t-1))] d\tau, \end{aligned}$$

by Lemma 2.1 and **(G)**-(ii) we have

$$(\partial_t^m \partial_x^\alpha \nabla^j G_\theta)(\cdot, t) \in \sum_{i=0}^{m'} \langle (\partial_t^{m+i} \partial_x^\alpha \nabla^j G_\theta)(\cdot, 1) \rangle + L_K \subset \mathcal{L}_{K,j} \subset L_{K'}, \quad t > 0,$$

for any $0 \leq K' \leq K$ with $K' < \theta$. This together with (2.19) implies assertion (b). Thus Theorem 1.2 follows. \square

Proof of Theorem 1.3. We can assume, without loss of generality, that $C(\varphi) = 0$, that is

$$\int_{\mathbf{R}^N} x \varphi(x) dx = 0. \quad (2.20)$$

We prove that $H(t) \neq \emptyset$ for any $t > 0$. For any $t > 0$, since

$$\int_{\mathbf{R}^N} [S_\theta(t)\varphi](x) dx = \int_{\mathbf{R}^N} \varphi(x) dx > 0,$$

we can find $x_t \in \mathbf{R}^N$ such that $[S_\theta(t)\varphi](x_t) > 0$. By (2.2) we can find $L > 0$ such that

$$\begin{aligned} [S_\theta(t)\varphi](x) &\leq Ct^{-\frac{N}{\theta}} \int_{\mathbf{R}^N} (1 + t^{-\frac{1}{\theta}}|x - y|)^{-N-\theta} |\varphi(y)| dy \\ &\leq Ct^{-\frac{N}{\theta}} \int_{B(0,L)} (1 + t^{-\frac{1}{\theta}}|x - y|)^{-N-\theta} |\varphi(y)| dy + Ct^{-\frac{N}{\theta}} \int_{\mathbf{R}^N \setminus B(0,L)} |\varphi(y)| dy \\ &\leq Ct^{-\frac{N}{\theta}} (1 + t^{-\frac{1}{\theta}}L)^{-N-\theta} \|\varphi\|_1 + Ct^{-\frac{N}{\theta}} \int_{\mathbf{R}^N \setminus B(0,L)} |\varphi(y)| dy \\ &< [S_\theta(t)\varphi](x_t) \end{aligned}$$

for $x \in \mathbf{R}^N \setminus B(0, 2L)$. This means that $H(t) \neq \emptyset$ for any $t > 0$.

We study the behavior of $H(t)$ as $t \rightarrow \infty$. It follows from (2.20) and Theorem 1.1 with $q = \infty$, $j = 0$ and $K = 1$ that

$$\|S_\theta(t)\varphi - M(\varphi)g_{0,\theta}(t)\|_\infty \leq Ct^{-\frac{N+1}{\theta}}, \quad t > 0. \quad (2.21)$$

For any $\epsilon > 0$, by **(G)**-(iii) and (2.21) we have

$$\begin{aligned} [S_\theta(t)\varphi](x) - [S_\theta(t)\varphi](0) &= M(\varphi)t^{-\frac{N}{\theta}} \left\{ G_\theta(t^{-\frac{1}{\theta}}x, 1) - G_\theta(0, 1) \right\} + O(t^{-\frac{N+1}{\theta}}) \\ &\leq -M(\varphi)t^{-\frac{N}{\theta}} \{G_\theta(0, 1) - G_\theta(\epsilon, 1)\} + O(t^{-\frac{N+1}{\theta}}) < 0 \end{aligned}$$

for all sufficiently large $t > 0$ uniformly on the set $\{x \in \mathbf{R}^N : |x| \geq \epsilon t^{\frac{1}{\theta}}\}$. This implies that

$$H(t) \subset B(0, \epsilon t^{\frac{1}{\theta}}) \quad (2.22)$$

for all sufficiently large $t > 0$.

By (2.20) we apply Theorem 1.1 with $q = \infty$, $j = 0$ and $K = 2$ to have

$$[S_\theta(t)\varphi](x) = M(\varphi)G_\theta(x, t) + \sum_{|\alpha|=2} c_\alpha g_{\alpha, \theta}(x, t) + o\left(t^{-\frac{N+2}{\theta}}\right) \quad (2.23)$$

as $t \rightarrow \infty$ uniformly for $x \in \mathbf{R}^N$, where

$$c_\alpha := \int_{\mathbf{R}^N} x^\alpha \varphi(x) dx.$$

Taking a sufficiently small $\epsilon > 0$ if necessary, by (1.7), (2.20) and (2.23) we obtain

$$\begin{aligned} & [S_\theta(t)\varphi](x) \\ &= M(\varphi)t^{-\frac{N}{\theta}}G_\theta(t^{-\frac{1}{\theta}}x, 1) + t^{-\frac{N+2}{\theta}} \sum_{|\alpha|=2} \frac{c_\alpha}{\alpha!} (\partial_x^\alpha G_\theta)(t^{-\frac{1}{\theta}}x, 1) + o\left(t^{-\frac{N+2}{\theta}}\right) \\ &= M(\varphi)t^{-\frac{N}{\theta}} \left[G_\theta(0, 1) + t^{-\frac{2}{\theta}} \sum_{|\alpha|=2} \frac{1}{\alpha!} (\partial_x^\alpha G_\theta)(0, 1) x^\alpha + O\left(t^{-\frac{3}{\theta}}|x|^3\right) \right] \\ &\quad + t^{-\frac{N+2}{\theta}} \sum_{|\alpha|=2} \frac{c_\alpha}{\alpha!} \left[(\partial_x^\alpha G_\theta)(0, 1) + O(t^{-\frac{1}{\theta}}|x|) \right] + o\left(t^{-\frac{N+2}{\theta}}\right) \end{aligned} \quad (2.24)$$

for all sufficiently large $t > 0$ uniformly on $\{x \in \mathbf{R}^N : |x| \leq \epsilon t^{\frac{1}{\theta}}\}$. On the other hand, by (2.1) we have

$$\frac{\partial^2 G_\theta}{\partial x_i^2}(0, 1) = -(2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \xi_i^2 e^{-|\xi|^\theta} d\xi < 0, \quad i = 1, \dots, N. \quad (2.25)$$

Taking a sufficiently small $\epsilon > 0$ if necessary again, by (2.24) and (2.25), for any $R > 0$, we obtain

$$\begin{aligned} & [S_\theta(t)\varphi](x) - [S_\theta(t)\varphi](0) \\ &= M(\varphi)t^{-\frac{N+2}{\theta}} \sum_{|\alpha|=2} \frac{1}{\alpha!} (\partial_x^\alpha G_\theta)(0, 1) x^\alpha + O(t^{-\frac{N+3}{\theta}}(|x| + |x|^3)) + o\left(t^{-\frac{N+2}{\theta}}\right) \\ &\leq -CM(\varphi)t^{-\frac{N+2}{\theta}}|x|^2 + O\left(\epsilon t^{-\frac{N+2}{\theta}}\right) + O\left(\epsilon t^{-\frac{N+2}{\theta}}|x|^2\right) + o\left(t^{-\frac{N+2}{\theta}}\right) \\ &\leq -CM(\varphi)t^{-\frac{N+2}{\theta}}|x|^2 + O\left(\epsilon t^{-\frac{N+2}{\theta}}\right) + o\left(t^{-\frac{N+2}{\theta}}\right) \\ &\leq -CM(\varphi)t^{-\frac{N+2}{\theta}}R^2 + O\left(\epsilon t^{-\frac{N+2}{\theta}}\right) + o\left(t^{-\frac{N+2}{\theta}}\right) \end{aligned} \quad (2.26)$$

for $x \in \mathbf{R}^N$ with $R \leq |x| \leq \epsilon t^{\frac{1}{\theta}}$ and sufficiently large $t > 0$. Since R is arbitrary, we deduce from (2.22) and (2.26) that

$$\lim_{t \rightarrow \infty} \sup_{x \in H(t)} |x| = 0. \quad (2.27)$$

Furthermore, by (2.1) and (2.25) we apply Theorem 1.1 with $q = \infty$, $j = 2$ and $K = 2$ to obtain

$$\begin{aligned} (\nabla^2[S_\theta(t)\varphi](x)y, y) &= M(\varphi)t^{-\frac{N+2}{\theta}}((\nabla^2 G_\theta)(0, 1)y, y) + o(t^{-\frac{N+2}{\theta}})|y|^2 \\ &\leq -CM(\varphi)t^{-\frac{N+2}{\theta}}|y|^2, \quad y \in \mathbf{R}^N, \end{aligned} \quad (2.28)$$

for $x \in \mathbf{R}^N$ with $|x| \leq \epsilon t^{\frac{1}{\theta}}$ and sufficiently large $t > 0$. By (2.27) and (2.28) we see that $H(t)$ consists of one point $x(t) \in \mathbf{R}^N$ for all sufficiently large $t > 0$. In addition, the implicit function theorem implies that $\{x(t)\}$ moves along a smooth curve in \mathbf{R}^N . Thus Theorem 1.3 follows. \square

3 Fractional diffusion equation with inhomogeneous term

Let $K \geq 0$. Let ψ be a radially symmetric smooth function in \mathbf{R}^N such that

$$\partial_x^\alpha \psi \in L^\infty \cap L_{K+|\alpha|} \text{ for } \alpha \in \mathbf{M}_K, \quad \int_{\mathbf{R}^N} \psi(y) dy = 1. \quad (3.1)$$

For any $\alpha \in \mathbf{M}$, $x \in \mathbf{R}^N$, $t > 0$ and $s \geq 0$, we define

$$\begin{aligned} \psi(x, s) &:= (1+s)^{-\frac{N}{\theta}} \psi\left((1+s)^{-\frac{1}{\theta}} x\right), \\ \psi_\alpha(x, s) &:= (1+s)^{-\frac{N+|\alpha|}{\theta}} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha \psi)\left((1+s)^{-\frac{1}{\theta}} x\right), \\ \Psi_{\alpha, \theta}(x, t : s) &:= \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha [S_\theta(t)\psi(\cdot, s)](x). \end{aligned} \quad (3.2)$$

It follows from (3.1) and (3.2) that

$$(1+s)^{\frac{N}{\theta}(1-\frac{1}{q})+\frac{|\alpha|}{\theta}} \|\psi_\alpha(s)\|_q + (1+s)^{-\frac{\ell}{\theta}+\frac{|\alpha|}{\theta}} \| |\psi_\alpha(s)| \|_\ell < +\infty, \quad s \geq 0, \quad (3.3)$$

for $1 \leq q \leq \infty$ and $0 \leq \ell < K + |\alpha|$. Furthermore,

$$\Psi_{\alpha, \theta}(x, t : s) = [S_\theta(t)\psi_\alpha(\cdot, s)](x) \quad (3.4)$$

for $\alpha \in \mathbf{M}_K$, $x \in \mathbf{R}^N$, $t > 0$ and $s \geq 0$. The precise description of the asymptotic behavior of $\Psi_{\alpha, \theta}(\cdot, t : s)$ as $t \rightarrow \infty$ can be given by Theorem 1.1 (see also Theorem 1.2).

In this section, by using $\{\Psi_{\alpha, \theta}\}$ we obtain the asymptotic expansions of the solution u to the fractional diffusion equation with an inhomogeneous term $F = F(x, t)$

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = F \quad \text{in } \mathbf{R}^N \times (0, \infty), \quad u(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}^N, \quad (3.5)$$

where $\varphi \in L^\infty \cap L_K$ and $F \in L^\infty(0, T : L_K)$ for any $T > 0$. Similarly to (1.2), we say that u is a solution of (3.5) if u is a continuous function in $\mathbf{R}^N \times (0, \infty)$ and u satisfies

$$\begin{aligned} u(x, t) &= \int_{\mathbf{R}^N} G_\theta(x-y, t) \varphi(y) dy + \int_0^t \int_{\mathbf{R}^N} G_\theta(x-y, t-s) F(y, s) dy ds \\ &\equiv [S_\theta(t)\varphi](x) + \int_0^t [S_\theta(t-s)F(\cdot, s)](x) ds, \quad x \in \mathbf{R}^N, \quad t > 0. \end{aligned}$$

We state the main result of this section, which is a generalization of [18, Theorems 1.1 and 1.2] (see Remark 3.1 (i)). In what follows, for any $f \in L_K$, $\alpha \in \mathbf{M}_K$ and $s \geq 0$, we define $M_\alpha(f, s)$ inductively in α by

$$\begin{cases} M_0(f, s) := \int_{\mathbf{R}^N} f(x) dx & \text{if } \alpha = 0, \\ M_\alpha(f, s) := \int_{\mathbf{R}^N} x^\alpha f(x) dx - \sum_{\beta \leq \alpha, \beta \neq \alpha} M_\beta(f, s) \int_{\mathbf{R}^N} x^\alpha \psi_\beta(x, s) dx & \text{if } \alpha \neq 0. \end{cases} \quad (3.6)$$

Then it follows from (3.1) and (3.6) that

$$\int_{\mathbf{R}^N} x^\beta \left[f(x) - \sum_{|\alpha| \leq K} M_\alpha(f, s) \psi_\alpha(x, s) \right] dx = 0 \quad (3.7)$$

for $\beta \in \mathbf{M}_K$ and $s \geq 0$.

Theorem 3.1 *Let $N \geq 1$, $0 < \theta < 2$ and $K \geq 0$.*

(i) *Let $\varphi \in L^\infty \cap L_K$ and set*

$$w_1(x, t) := \sum_{|\alpha| \leq K} M_\alpha(\varphi, 0) \Psi_{\alpha, \theta}(x, t : 0).$$

Then there exists a constant $C_1 > 0$ such that

$$\begin{aligned} (1+t)^{\frac{N}{\theta}(1-\frac{1}{q})} \|S_\theta(t)\varphi - w_1(t)\|_q + (1+t)^{-\frac{\ell}{\theta}} \| \|S_\theta(t)\varphi - w_1(t)\| \|_\ell \\ \leq C_1 (1+t)^{-\frac{K}{\theta}} (\|\varphi\|_\infty + \|\varphi\|_1 + \| \|\varphi\| \|_K) \end{aligned} \quad (3.8)$$

for $t > 0$, $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$. In particular, for any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$,

$$\lim_{t \rightarrow \infty} t^{\frac{K}{\theta}} \left[t^{\frac{N}{\theta}(1-\frac{1}{q})} \|S_\theta(t)\varphi - w_1(t)\|_q + t^{-\frac{\ell}{\theta}} \| \|S_\theta(t)\varphi - w_1(t)\| \|_\ell \right] = 0. \quad (3.9)$$

(ii) *Let $1 \leq q \leq \infty$ and let F be a measurable function in $\mathbf{R}^N \times (0, \infty)$ such that*

$$\begin{aligned} E_{K,q}[F](t) := (1+t)^{\frac{K}{\theta}} \left[(1+t)^{\frac{N}{\theta}(1-\frac{1}{q})} \|F(t)\|_q + \|F(t)\|_1 \right] + \| \|F(t)\| \|_K \\ \in L^\infty(0, T) \end{aligned} \quad (3.10)$$

for any $T > 0$. Set

$$\begin{aligned} v(x, t) &:= \int_0^t [S_\theta(t-s)F(\cdot, s)](x) ds, \\ w_2(x, t) &:= \sum_{|\alpha| \leq K} \int_0^t M_\alpha(F(s), s) \Psi_{\alpha, \theta}(x, t-s : s) ds. \end{aligned}$$

Then there exists a constant $C_2 > 0$ such that

$$\|v(t) - w_2(t)\|_q + |||v(t) - w_2(t)|||_\ell \leq C_2 \int_0^t E_{K,q}[F](s) ds, \quad t > 0, \quad (3.11)$$

for $0 \leq \ell \leq K$. Furthermore, there exists a constant $C_3 > 0$ such that, for any $\epsilon > 0$ and $T > 0$,

$$\begin{aligned} t^{\frac{N}{\theta}(1-\frac{1}{q})} \|v(t) - w_2(t)\|_q + t^{-\frac{\ell}{\theta}} |||v(t) - w_2(t)|||_\ell \\ \leq \epsilon t^{-\frac{K}{\theta}} + C_3 t^{-\frac{K}{\theta}} \int_T^t E_{K,q}[F](s) ds \end{aligned} \quad (3.12)$$

for all sufficiently large $t > T$ and all $0 \leq \ell \leq K$. In particular, if

$$\int_0^\infty E_{K,q}[F](s) ds < \infty,$$

then

$$\lim_{t \rightarrow \infty} t^{\frac{K}{\theta}} \left[t^{\frac{N}{\theta}(1-\frac{1}{q})} \|v(t) - w_2(t)\|_q + t^{-\frac{\ell}{\theta}} |||v(t) - w_2(t)|||_\ell \right] = 0. \quad (3.13)$$

The constants C_1 , C_2 and C_3 depend only on N , θ and K .

We first prove assertion (i) of Theorem 3.1.

Proof of Theorem 3.1 (i). Let $K \geq 0$ and $\varphi \in L^\infty \cap L_K$. Put

$$\tilde{\varphi}(x) := \varphi(x) - \sum_{|\alpha| \leq K} M_\alpha(\varphi, 0) \psi_\alpha(x, 0). \quad (3.14)$$

By (3.7) we see that

$$\int_{\mathbf{R}^N} x^\beta \tilde{\varphi}(x) dx = 0 \quad \text{for all } \beta \in \mathbf{M}_K.$$

This together with Theorem 1.1, (3.3) and (3.14) implies

$$\sup_{t>0} t^{\frac{K}{\theta}} \left[t^{\frac{N}{\theta}(1-\frac{1}{q})} \|S_\theta(t) \tilde{\varphi}\|_q + t^{-\frac{\ell}{\theta}} |||S_\theta(t) \tilde{\varphi}|||_\ell \right] \quad (3.15)$$

$$\leq C |||\tilde{\varphi}|||_K \leq C(\|\varphi\|_1 + |||\varphi|||_K),$$

$$\lim_{t \rightarrow \infty} t^{\frac{K}{\theta}} \left[t^{\frac{N}{\theta}(1-\frac{1}{q})} \|S_\theta(t) \tilde{\varphi}\|_q + t^{-\frac{\ell}{\theta}} |||S_\theta(t) \tilde{\varphi}|||_\ell \right] = 0, \quad (3.16)$$

for $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$. On the other hand, it follows from (3.4) and (3.14) that

$$[S_\theta(t) \tilde{\varphi}](x) = [S_\theta(t) \varphi](x) - w_1(x, t), \quad x \in \mathbf{R}^N, \quad t > 0. \quad (3.17)$$

By (2.4), (3.1), (3.2), (3.3) and (3.17) we obtain

$$\begin{aligned} \|S_\theta(t)\tilde{\varphi}\|_q &\leq \|S_\theta(t)\varphi\|_q + \sum_{|\alpha|\leq K} |M_\alpha(\varphi, 0)| \|\Psi_{\alpha, \theta}(t : 0)\|_q \\ &\leq C(\|\varphi\|_\infty + \|\varphi\|_1 + \|\varphi\|_K), \quad t > 0, \end{aligned} \quad (3.18)$$

for $1 \leq q \leq \infty$. This together with (3.15) implies

$$\begin{aligned} |||S_\theta(t)\tilde{\varphi}|||_\ell &= (1+t)^{\frac{\ell}{\theta}} \int_{\mathbf{R}^N} \left(\frac{|x|}{(1+t)^{\frac{1}{\theta}}} \right)^\ell |[S_\theta(t)\tilde{\varphi}](x)| \, dx \\ &\leq C(1+t)^{\frac{\ell}{\theta}} \int_{\mathbf{R}^N} \left[1 + \left(\frac{|x|}{(1+t)^{\frac{1}{\theta}}} \right)^K \right] |[S_\theta(t)\tilde{\varphi}](x)| \, dx \\ &= C(1+t)^{\frac{\ell}{\theta}} \|S_\theta(t)\tilde{\varphi}\|_1 + C(1+t)^{\frac{\ell-K}{\theta}} |||S_\theta(t)\tilde{\varphi}|||_K \\ &\leq C(1+t)^{\frac{\ell}{\theta}} (\|\varphi\|_\infty + \|\varphi\|_1 + \|\varphi\|_K), \quad t > 0, \end{aligned} \quad (3.19)$$

for $0 \leq \ell \leq K$. Therefore, by (3.15), (3.16), (3.17), (3.18) and (3.19) we obtain (3.8) and (3.9). Thus Theorem 3.1 (i) follows. \square

For the proof of Theorem 3.1 (ii), we prepare the following lemma, which is proved by a similar argument as in [18, Lemma 2.2] with the aid of (3.3).

Lemma 3.1 *Assume the same assumptions as in Theorem 3.1 (ii). Let $1 \leq r \leq q \leq \infty$ and $0 \leq \ell \leq K$. Set*

$$\tilde{F}(x, t) := F(x, t) - \sum_{|\alpha|\leq K} M_\alpha(F(t), t) \psi_\alpha(x, t). \quad (3.20)$$

Then there exists a constant C such that

$$\begin{aligned} \|\tilde{F}(t)\|_r &\leq C(1+t)^{-\frac{N}{\theta}(1-\frac{1}{r})-\frac{K}{\theta}} E_{K,q}[F](t), \\ |||\tilde{F}(t)|||_\ell &\leq C(1+t)^{-\frac{K-\ell}{\theta}} E_{K,q}[F](t), \end{aligned}$$

for $t > 0$.

Proof of Theorem 3.1 (ii). Let $0 \leq \ell \leq K$ and $1 \leq q \leq \infty$. It follows from (3.4) and (3.20) that

$$\begin{aligned} v(x, t) - w_2(x, t) &= \int_0^t [S_\theta(t-s)\tilde{F}(\cdot, s)](x) \, ds \\ &= \int_{t/2}^t [S_\theta(t-s)\tilde{F}(\cdot, s)](x) \, ds + \int_0^{t/2} [S_\theta(t-s)\tilde{F}(\cdot, s)](x) \, ds \\ &=: I_1(x, t) + I_2(x, t) \end{aligned} \quad (3.21)$$

for $(x, t) \in \mathbf{R}^N \times (0, \infty)$. Furthermore, by (3.7) we see that

$$\int_{\mathbf{R}^N} x^\beta \tilde{F}(x, s) dx = 0 \quad \text{for all } \beta \in \mathbf{M}_K, s > 0. \quad (3.22)$$

This together with Theorem 1.1 (a) with $K = \ell$, Lemma 3.1 and (2.4) implies that

$$\begin{aligned} & \|v(t) - w_2(t)\|_q + \| \|v(t) - w_2(t)\| \|_\ell \\ & \leq C \int_0^t \|\tilde{F}(s)\|_q ds + C \int_0^t \| \|\tilde{F}(s)\| \|_\ell ds \leq C \int_0^t E_{K,q}[F](s) ds \end{aligned}$$

for $t > 0$. This implies (3.11). Similarly, we have

$$\begin{aligned} & t^{\frac{N}{\theta}(1-\frac{1}{q})} \|I_1(t)\|_q + (1+t)^{-\frac{\ell}{\theta}} \| \|I_1(t)\| \|_\ell \\ & \leq C t^{\frac{N}{\theta}(1-\frac{1}{q})} \int_{t/2}^t \|\tilde{F}(s)\|_q ds + C(1+t)^{-\frac{\ell}{\theta}} \int_{t/2}^t \| \|\tilde{F}(s)\| \|_\ell ds \\ & \leq C t^{-\frac{K}{\theta}} \int_{t/2}^t \| \|\tilde{F}(s)\| \|_K ds \leq C t^{-\frac{K}{\theta}} \int_{t/2}^t E_{K,q}[F](s) ds. \end{aligned} \quad (3.23)$$

Let $T > 0$. By Theorem 1.1, Lemma 3.1 and (3.22) we apply the Lebesgue dominated convergence theorem to obtain

$$\limsup_{t \rightarrow \infty} t^{\frac{K}{\theta}} \int_0^T \left[t^{\frac{N}{\theta}(1-\frac{1}{q})} \|S_\theta(t-s)\tilde{F}(\cdot, s)\|_q + t^{-\frac{\ell}{\theta}} \| \|S_\theta(t-s)\tilde{F}(\cdot, s)\| \|_\ell \right] ds = 0. \quad (3.24)$$

Furthermore, by Theorem 1.1 (a), Lemma 3.1 and (3.22) we obtain

$$\begin{aligned} & t^{\frac{N}{\theta}(1-\frac{1}{q})} \int_T^{t/2} \|S_\theta(t-s)\tilde{F}(\cdot, s)\|_q ds + t^{-\frac{\ell}{\theta}} \int_T^{t/2} \| \|S_\theta(t-s)\tilde{F}(\cdot, s)\| \|_\ell ds \\ & \leq C t^{\frac{N}{\theta}(1-\frac{1}{q})} \int_T^{t/2} (t-s)^{-\frac{N}{\theta}(1-\frac{1}{q})-\frac{K}{\theta}} \| \|\tilde{F}(s)\| \|_K ds \\ & \quad + C t^{-\frac{\ell}{\theta}} \int_T^{t/2} (t-s)^{-\frac{K-\ell}{\theta}} \| \|\tilde{F}(s)\| \|_K ds \\ & \leq C t^{-\frac{K}{\theta}} \int_T^{t/2} E_{K,q}[F](s) ds \end{aligned} \quad (3.25)$$

for $t \geq 2T$. Then, for any $\epsilon > 0$, we deduce from (3.24) and (3.25) that

$$t^{\frac{N}{\theta}(1-\frac{1}{q})} \|I_2(t)\|_q + t^{-\frac{\ell}{\theta}} \| \|I_2(t)\| \|_\ell \leq \epsilon t^{-\frac{K}{\theta}} + C t^{-\frac{K}{\theta}} \int_T^{t/2} E_{K,q}[F](s) ds \quad (3.26)$$

for all sufficiently large t . Then, by (3.21), (3.23) and (3.26) we obtain (3.12). Furthermore, (3.13) immediately follows from (3.12). Therefore the proof of Theorem 3.1 (ii) is complete. Thus Theorem 3.1 follows. \square

Remark 3.1 (i) Let $0 \leq K < \theta$ and set $\psi(x) = G_\theta(x, 1)$. It follows that $\psi(x, s) = G_\theta(x, 1 + s)$ and

$$\Psi_{\alpha, \theta}(x, t : s) = \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha [S_\theta(t) G_\theta(\cdot, 1 + s)](x) = g_{\alpha, \theta}(x, 1 + s + t)$$

for $x \in \mathbf{R}^N$, $t > 0$ and $s \geq 0$. Furthermore, the functions w_1 and w_2 in Theorem 3.1 are represented by

$$\begin{aligned} w_1(x, t) &= \sum_{|\alpha| \leq K} M_\alpha(\varphi, 0) g_{\alpha, \theta}(x, t + 1), \\ w_2(x, t) &= \sum_{|\alpha| \leq K} \left[\int_0^t M_\alpha(F(s), s) ds \right] g_{\alpha, \theta}(x, t + 1), \end{aligned}$$

respectively. Then, by Theorem 3.1 we obtain the same conclusions as in Theorems 1.1 and 1.2 in [18].

(ii) In the case $K \geq \theta$, since $G_\theta(\cdot, 1) \notin L_K$, the argument in Remark 3.1 (i) is not applicable.

(iii) It follows from (3.1) and (3.2) that $\psi_\alpha \in L^\infty \cap L_K$. This fact enables us to obtain a better asymptotic expansion w_1 of $S_\theta(t)\varphi$ than

$$\sum_{|\alpha| \leq K} \left[\int_{\mathbf{R}^N} x^\alpha \varphi(x) dx \right] g_{\alpha, \theta}(x, t)$$

(compare (3.8) with (1.10)) and to define the function w_2 in assertion (ii) of Theorem 3.1.

4 Nonlinear fractional diffusion equation

Let $K \geq 0$. Assume that a radially symmetric smooth function ψ satisfies (3.1) and

$$|(\partial_x^\alpha \psi)(x)| \leq C G_\theta(x, 1), \quad \alpha \in \mathbf{M}_K, \quad (4.1)$$

for $x \in \mathbf{R}^N$. We apply the arguments in the previous sections to study the asymptotic expansions of the solution u of (1.1) satisfying (1.4). Let

$$\mathcal{L}_K^* := L_K + \left\{ \phi \in L^\infty : \operatorname{ess\,sup}_{x \in \mathbf{R}^N} (1 + |x|)^{N+\theta} |\phi(x)| < \infty \right\}.$$

Theorem 4.1 Let $N \geq 1$, $0 < \theta < 2$, $p > 1 + \theta/N$ and $K \geq 0$. Assume $\varphi \in L^\infty \cap \mathcal{L}_K^*$. Let u be a global in time solution of (1.1) such that

$$\sup_{t>0} (1+t)^{\frac{N}{\theta}} \|u(t)\|_\infty < \infty. \quad (4.2)$$

Futhermore, let ψ be a radially symmetric smooth function in \mathbf{R}^N satisfying (3.1) and (4.1). Define

$$U_0(x, t) := [S_\theta(t)\varphi](x) + \sum_{|\alpha| \leq K} \int_0^t M_\alpha(F(s), s) \Psi_{\alpha, \theta}(x, t-s : s) ds,$$

where $M_\alpha(F(s), s)$ and $\Psi_{\alpha, \theta}(x, t-s : s)$ are as in Section 3. Assume $K + N < p(N + \theta)$. Then, for any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$,

$$\sup_{t>0} \left[(1+t)^{\frac{N}{\theta}(1-\frac{1}{q})} \|u(t) - U_0(t)\|_q + (1+t)^{-\frac{\ell}{\theta}} \| |u(t) - U_0(t)| \|_\ell \right] < \infty \quad (4.3)$$

and

$$\begin{aligned} & t^{\frac{N}{\theta}(1-\frac{1}{q})} \|u(t) - U_0(t)\|_q + t^{-\frac{\ell}{\theta}} \| |u(t) - U_0(t)| \|_\ell \\ &= \begin{cases} o(t^{-\frac{K}{\theta}}) + O(t^{-A_p+1}) & \text{if } A_p - 1 \neq K/\theta, \\ O(t^{-\frac{K}{\theta}} \log t) & \text{if } A_p - 1 = K/\theta, \end{cases} \end{aligned} \quad (4.4)$$

as $t \rightarrow \infty$, where $A_p := N(p-1)/\theta > 1$.

For the proof of Theorem 4.1, we prepare the following two lemmas on \mathcal{L}_K^* .

Lemma 4.1 *Let $0 < \theta < 2$ and $K \geq 0$. Then*

- (a) $\mathcal{L}_{K,0} \subset \mathcal{L}_K^*$;
- (b) $\bigcup_{t>0} \{S_\theta(t)\varphi : \varphi \in \mathcal{L}_K^*\} \subset \mathcal{L}_K^*$;
- (c) *Let $\varphi \in \mathcal{L}_K^*$ and let ϕ be a measurable function in \mathbf{R}^N such that $|\phi(x)| \leq |\varphi(x)|$ for almost all $x \in \mathbf{R}^N$. Then $\phi \in \mathcal{L}_K^*$.*

Proof. Assertion (a) follows from Lemma 2.1. Let $\varphi \in \mathcal{L}_K^*$. Then we can find $\varphi_1 \in L_K$ and $\varphi_2 \in L^\infty$ with

$$\operatorname{ess\,sup}_{x \in \mathbf{R}^N} (1+|x|)^{N+\theta} |\varphi_2(x)| < \infty \quad (4.5)$$

such that $\varphi = \varphi_1 + \varphi_2$.

We prove assertion (b). It follows from Theorem 1.2 (b) that

$$\{S_\theta(t)\varphi_1 : t > 0\} \subset \mathcal{L}_{K,0}. \quad (4.6)$$

By (G)-(iii) and (4.5) we have

$$|\varphi_2(x)| \leq CG_\theta(x, 1)$$

for almost all $x \in \mathbf{R}^N$. Then

$$|[S_\theta(t)\varphi_2](x)| \leq [S_\theta(t)|\varphi_2|](x) \leq CG_\theta(x, 1+t), \quad x \in \mathbf{R}^N, \quad t > 0,$$

which together with (2.2) implies that

$$\sup_{x \in \mathbf{R}^N} (1 + |x|)^{N+\theta} |[S_\theta(t)\varphi_2](x)| < \infty. \quad (4.7)$$

Therefore, by assertion (a), (4.6) and (4.7) we obtain $\{S_\theta(t)\varphi : \varphi \in \mathcal{L}_K^*\} \subset \mathcal{L}_K^*$ for any $t > 0$. Thus assertion (b) follows.

We prove assertion (c). Set

$$\phi^\pm := \max\{\pm\phi, 0\}, \quad E^\pm := \{x \in \mathbf{R}^N : \phi^\pm(x) > |\varphi_1(x)|\}.$$

Since $|\phi(x)| \leq |\varphi(x)| \leq |\varphi_1(x)| + |\varphi_2(x)|$ for almost all $x \in \mathbf{R}^N$, we see that

$$0 \leq (\phi^\pm(x) - |\varphi_1(x)|)\chi_{E^\pm}(x) \leq |\varphi_2(x)| \leq C(1 + |x|)^{-N-\theta} \quad (4.8)$$

for almost all $x \in \mathbf{R}^N$, where χ_{E^\pm} are the characteristic functions of E^\pm , respectively. Furthermore, since $|\varphi_1|\chi_{E^\pm} \leq |\varphi_1|$ and $\phi^\pm\chi_{\mathbf{R}^N \setminus E^\pm} \leq |\varphi_1|$, we have

$$|\varphi_1|\chi_{E^\pm} + \phi^\pm\chi_{\mathbf{R}^N \setminus E^\pm} \in L_K. \quad (4.9)$$

Therefore, by (4.8) and (4.9) we obtain

$$\phi^\pm = (\phi^\pm - |\varphi_1|)\chi_{E^\pm} + |\varphi_1|\chi_{E^\pm} + \phi^\pm\chi_{\mathbf{R}^N \setminus E^\pm} \in \mathcal{L}_K^*.$$

This implies that $\phi \in \mathcal{L}_K^*$ and assertion (c) follows. Therefore the proof of Lemma 4.1 is complete. \square

Lemma 4.2 *Let u be a global in time solution of (1.1) with $\varphi \in L^\infty \cap \mathcal{L}_K^*$ and $p > 1 + \theta/N$. Assume (4.2). Then*

$$u(t) \in L^\infty \cap \mathcal{L}_K^*, \quad t > 0. \quad (4.10)$$

Furthermore, if $K + N < p(N + \theta)$, then there exists a constant C such that

$$|||u(t)^p|||_K \leq C(1 + t)^{-A_p + \frac{K}{\theta}}, \quad t > 0. \quad (4.11)$$

Proof. We prove (4.10). It follows from (4.2) that

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u \leq C(1 + t)^{-A_p} u \quad \text{in } \mathbf{R}^N \times (0, \infty).$$

Since $A_p > 1$, the comparison principle implies that

$$|u(x, t)| \leq C[S_\theta(t)|\varphi|](x), \quad x \in \mathbf{R}^N, \quad t > 0. \quad (4.12)$$

It follows from $\varphi \in \mathcal{L}_K^*$ that $|\varphi| \in \mathcal{L}_K^*$. This together with Lemma 4.1 (b) yields $S_\theta(t)|\varphi| \in \mathcal{L}_K^*$. Then, by Lemma 4.1 (c) and (4.12) we see that $u(t) \in \mathcal{L}_K^*$ for $t > 0$. This together with (4.2) implies (4.10).

We prove (4.11). We assume that $K+N < p(N+\theta)$. By (2.2) we see that $G_\theta(\cdot, t)^p \in L_K$ and

$$|||G_\theta(t)^p|||_K \leq Ct^{-A_p + \frac{K}{\theta}}, \quad t > 0. \quad (4.13)$$

Let $\varphi \in L^\infty \cap \mathcal{L}_K^*$. Then we can find $\varphi_1 \in L^\infty \cap L_K$ and $\varphi_2 \in L^\infty$ with

$$|\varphi_2(x)| \leq CG_\theta(x, 1) \quad \text{for almost all } x \in \mathbf{R}^N, \quad (4.14)$$

such that $\varphi = \varphi_1 + \varphi_2$. By $\varphi_1 \in L_K$ we define

$$v_1(x, t) := [S_\theta(t)|\varphi_1|](x) - \sum_{|\alpha| \leq K} M_\alpha(|\varphi_1|, 0)\Psi_{\alpha, \theta}(x, t : 0), \quad (4.15)$$

where $\Psi_{\alpha, \theta}$ is the function given by (3.2). Then Theorem 3.1 (i) implies that

$$|||v_1(t)^p|||_K \leq \|v_1(t)\|_\infty^{p-1} |||v_1(t)|||_K \leq C(1+t)^{-A_p - \frac{K(p-1)}{\theta}}, \quad t > 0. \quad (4.16)$$

On the other hand, by (3.4) and (4.1) we have

$$\begin{aligned} \Psi_{\alpha, \theta}(x, t : 0) &= [S_\theta(t)\psi_\alpha(0)](x) \leq [S_\theta(t)|\psi_\alpha(0)|](x) \\ &\leq C[S_\theta(t)G_\theta(\cdot, 1)](x) \leq CG_\theta(x, t+1) \end{aligned}$$

for $x \in \mathbf{R}^N$, $t > 0$ and $\alpha \in \mathbf{M}_K$. This together with (4.13), (4.15) and (4.16) yields

$$\begin{aligned} |||(S_\theta(t)|\varphi_1|)^p|||_K &\leq C|||v_1(t)^p|||_K + C \sum_{|\alpha| \leq K} |||\Psi_{\alpha, \theta}(t : 0)^p|||_K \\ &\leq C(1+t)^{-A_p - \frac{K(p-1)}{\theta}} + C|||G_\theta(t+1)^p|||_K \leq C(1+t)^{-A_p + \frac{K}{\theta}} \end{aligned} \quad (4.17)$$

for $t > 0$. Furthermore, by (4.13) and (4.14) we obtain

$$|||(S_\theta(t)|\varphi_2|)^p|||_K \leq C|||[S_\theta(t)G_\theta(1)]^p|||_K = C|||G_\theta(t+1)^p|||_K \leq C(1+t)^{-A_p + \frac{K}{\theta}} \quad (4.18)$$

for $t > 0$. On the other hand, by (4.12) we see that

$$|u(x, t)|^p \leq C[S_\theta(t)|\varphi|](x)^p \leq C[S_\theta(t)|\varphi_1|](x)^p + C[S_\theta(t)|\varphi_2|](x)^p$$

for $x \in \mathbf{R}^N$ and $t > 0$. This together with (4.17) and (4.18) implies that

$$|||u(t)^p|||_K \leq C(1+t)^{-A_p + \frac{K}{\theta}}, \quad t > 0.$$

Therefore we obtain (4.11), and the proof of Lemma 4.2 is complete. \square

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$. Since $\varphi \in \mathcal{L}_K^*$, by (2.4), (4.2) and (4.12) we have

$$\|F(u(t))\|_q \leq \|u(t)\|_\infty^{p-1} \|u(t)\|_q \leq C(1+t)^{-A_p - \frac{N}{\theta}(1-\frac{1}{q})}, \quad t > 0. \quad (4.19)$$

Then, by (3.10), (4.11) and (4.19) we have

$$E_{K,q}[F](t) \leq C(1+t)^{-A_p+\frac{K}{\theta}}, \quad t > 0. \quad (4.20)$$

On the other hand, it follows from Theorem 3.1 (ii) that

$$\|u(t) - U_0(t)\|_q + \|u(t) - U_0(t)\|_\ell \leq C \int_0^t E_{K,q}[F](s) ds, \quad t > 0.$$

Furthermore, for any $T > 0$ and $\epsilon > 0$, we obtain

$$t^{\frac{N}{\theta}(1-\frac{1}{q})} \|u(t) - U_0(t)\|_q + t^{-\frac{\ell}{\theta}} \|u(t) - U_0(t)\|_\ell \leq \epsilon t^{-\frac{K}{\theta}} + C_* t^{-\frac{K}{\theta}} \int_T^t E_{K,q}[F](s) ds$$

for all sufficiently large $t > T$, where C_* is a constant independent of T and ϵ . These together with (4.20) imply (4.3) and (4.4). Thus Theorem 4.1 follows. \square

Finally, combining the arguments in [15, Theorem 3.1], we obtain the following theorem.

Theorem 4.2 *Assume the same conditions as in Theorem 4.1. Define $U_n = U_n(x, t)$ ($n = 1, 2, \dots$) inductively by*

$$\begin{aligned} U_n(x, t) &:= U_0(x, t) + \int_0^t [S_\theta(t-s)F_{n-1}(\cdot, s)](x) ds \\ &\quad - \sum_{|\alpha| \leq K} \int_0^t M_\alpha(F_{n-1}(s), s) \Psi_{\alpha, \theta}(x, t-s : s) ds, \end{aligned} \quad (4.21)$$

where $n = 1, 2, \dots$ and $F_{n-1}(x, t) := F(U_{n-1}(x, t))$. Then, for any $1 \leq q \leq \infty$, $0 \leq \ell \leq K$, and $n = 0, 1, \dots$,

$$\sup_{t>0} \left\{ (1+t)^{\frac{N}{\theta}(1-\frac{1}{q})} \|u(t) - U_n(t)\|_q + (1+t)^{-\frac{K}{\theta}} \|u(t) - U_n(t)\|_K \right\} < \infty \quad (4.22)$$

and

$$\begin{aligned} &t^{\frac{N}{\theta}(1-\frac{1}{q})} \|u(t) - U_n(t)\|_q + t^{-\frac{\ell}{\theta}} \|u(t) - U_n(t)\|_\ell \\ &= \begin{cases} o(t^{-\frac{K}{\theta}}) + O(t^{-(n+1)(A_p-1)}) & \text{if } (n+1)(A_p-1) \neq K/\theta, \\ O(t^{-\frac{K}{\theta}} \log t) & \text{if } (n+1)(A_p-1) = K/\theta, \end{cases} \end{aligned} \quad (4.23)$$

as $t \rightarrow \infty$.

Proof of Theorem 4.2. By Theorem 4.1 we have (4.22) and (4.23) with $n = 0$. Assume that (4.22) and (4.23) hold for some $n = m \in \{0, 1, 2, \dots\}$. Set

$$\hat{F}(x, t) := F(u(x, t)) - F(U_m(x, t)).$$

Then it follows from (4.21) that

$$u(x, t) - U_{m+1}(x, t) = \hat{I}(x, t) \quad (4.24)$$

for $x \in \mathbf{R}^N$ and $t > 0$, where

$$\hat{I}(x, t) := \int_0^t [S_\theta(t-s)\hat{F}(\cdot, s)](x) ds - \sum_{|\alpha| \leq K} \int_0^t M_\alpha(\hat{F}(s), s) \Psi_{\alpha, \theta}(x, t-s) ds.$$

By (4.2) and (4.22) we apply the mean value theorem to obtain

$$E_{K,q}[\hat{F}](t) \leq C(1+t)^{-A_p} E_{K,q}[u - U_m](t), \quad t > 0,$$

where $1 \leq q \leq \infty$. This together with (3.10), (4.22) and (4.23) with $n = m$ implies that

$$E_{K,q}[\hat{F}](t) \leq C(1+t)^{-A_p + \frac{K}{\theta}}, \quad t > 0,$$

and

$$E_{K,q}[\hat{F}](t) = \begin{cases} o(t^{-A_p}) + O(t^{-A_p - (m+1)(A_p-1) + \frac{K}{\theta}}) & \text{if } (m+1)(A_p-1) \neq K/\theta, \\ O(t^{-A_p} \log t) & \text{if } (m+1)(A_p-1) = K/\theta, \end{cases}$$

as $t \rightarrow \infty$. Then we deduce from Theorem 3.1 (ii) that

$$\|\hat{I}(t)\|_q + |||\hat{I}(t)|||_\ell \leq C \int_0^t (1+s)^{-A_p + \frac{K}{\theta}} ds, \quad t > 0 \quad (4.25)$$

and

$$\begin{aligned} & t^{\frac{N}{\theta}(1-\frac{1}{q})} \|\hat{I}(t)\|_q + t^{-\frac{\ell}{\theta}} |||\hat{I}(t)|||_\ell \\ &= \begin{cases} o(t^{-\frac{K}{\theta}}) + O(t^{-(m+2)(A_p-1)}) & \text{if } (m+2)(A_p-1) \neq K/\theta, \\ O(t^{-\frac{K}{\theta}} \log t) & \text{if } (m+2)(A_p-1) = K/\theta, \end{cases} \end{aligned} \quad (4.26)$$

as $t \rightarrow \infty$, for any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$. Therefore, by (4.24), (4.25) and (4.26) we obtain (4.22) and (4.23) with $n = m+1$. This means that (4.22) and (4.23) hold for all $n = 0, 1, \dots$. Thus Theorem 4.2 follows. \square

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